Buckling of Segments of Toroidal Shells

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Nonlinear differential equations of equilibrium and buckling equations are derived for segments of toroidal shells near the equator and for segments near the crown. The equations are derived for shallow segments by including appropriate prescribed initial displacements in the nonlinear, flat-plate, strain-displacement equations and by varying the total potential energy of the system. Closed-form solutions to the buckling equations are obtained for simply supported segments near the equator, having either positive or negative Gaussian curvature under pressure loading with various inplane support conditions. Results are presented in the form of charts showing buckling coefficients as a function of a curvature parameter associated with the girth of the shell and a parameter associated with the ratio of principal curvatures. In many instances, the results indicate significant deviations in buckling stress for the toroidal shells over the buckling stress for the corresponding circular cylindrical shell under similar loading and support conditions.

Nomenclature

a	= radius of curvature (Figs. 1 and 2)
A, B, C	= const
D	= flexural stiffness of shell wall, $Et^3/12(1-\mu^2)$
E	= Young's modulus
k	= buckling coefficient, $-prl^2/D\pi^2$
l	= length of shell
m, n	= integers
N_x, N_y, N_{xy}	= stress resultants in rectangular coordinates
$N_\rho, N_\theta, N_{\rho\theta}$	= stress resultants in cylindrical coordinates
p	= lateral pressure, positive in positive w direction
ρ, θ, z	= cylindrical coordinates
r	= radius of shell equator (Fig. 1)
R	= central radius of toroidal segment near crown (Fig. 2)
t	= shell wall thickness
u, v, w	= displacements, tangential and normal (outward) to shell neutral surface
$\bar{u}, \bar{v}$	= displacements in x and y directions
w_0	= initial deflection
x, y, z	= rectangular coordinates
Z	= curvature parameter, $(l^2/rt)(1-\mu^2)^{1/2}$
α	= curvature parameter, r/a
β	= buckle wavelength parameter, $nl/\pi r$
$\epsilon_x, \epsilon_y, \gamma_{xy}$	= direct strains and shearing strain in rectangular coordinates
$\epsilon_\rho, \epsilon_\theta, \gamma_{\rho\theta}$	= direct strains and shearing strain in cylindrical coordinates
μ	= Poisson's ratio
Π	= total potential energy of shell
∇^4	= $\nabla^2\nabla^2$ where ∇^2 is Laplacian operator in two dimensions

Subscripts

A	= prebuckling displacements and stress resultants
B	= buckling displacements and stress resultants
$( )'$	= partial differentiation with respect to subscripts following comma

Introduction

SHELLS of double curvature are common in aerospace vehicle structures, and buckling is often an important design consideration for such shells. In this paper, nonlinear differential equations of equilibrium and buckling equations are derived for segments of toroidal shells, a type of double curvature shell that is attracting considerable interest at the present time. Solutions to the buckling equations are obtained for segments of toroidal shells near the equator, having either positive or negative Gaussian curvature (Fig. 1) subjected to various pressure loadings.

The nonlinear equilibrium equations are derived for shallow shell segments by including appropriate prescribed initial displacements in the nonlinear, flat-plate, strain-displacement relations and by varying the total potential energy of the system. These equations reduce to the large-deflection Donnell equations for the case of a circular cylindrical shell and to the Marguerre large-deflection equations for the case of a shallow spherical cap. The buckling equations are derived from the nonlinear equations in a rigorous manner by assuming the changes, which occur at buckling to be small. One set of equations is obtained which is applicable to segments of a toroidal shell near the crown, and another set is obtained which is applicable to segments near the equator. For segments near the equator, the classical assumption of constant deflection prior to buckling leads to buckling equations, which are the same as those given by Becker in Ref. 1 for shells of double curvature having constant, but not necessarily equal, principal curvatures.

Closed-form solutions are presented to the buckling equations (obtained by using the classical assumption) for segments of toroidal shells near the equator subjected to lateral pressure. The assumed edge support conditions are simple support either with zero edge displacement, with hydrostatic pressure loaded edges, or with freedom for over-all edge extension in the axial direction. Results are presented in the form of charts showing buckling stress coefficient as a function of a curvature parameter associated with the girth of the shell and a parameter associated with the ratio of principal curvatures. The results indicate significant deviations in buckling stress for the toroidal shells over the buckling stress for the corresponding circular cylindrical shell under similar loading and support conditions.

For many buckling problems involving deep shells, shallow shell analysis should give engineering estimates to over-all buckling loads. In the present paper, an estimate of the external buckling pressure for a complete torus is obtained on the basis of the study of the shallow segment near the outer equator.

Nonlinear Differential Equations of Equilibrium

In this section, the nonlinear differential equations of equilibrium are derived for shallow segments of a torus near the equators and near the crown. For segments near the equators, the equations are derived in the rectangular coordinates of an osculating plane. For segments near the

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crown, the equations are derived in plane polar coordinates. In both cases, the equations are derived from the strains of nonlinear flat-plate theory including initial deflections (see Ref. 2) using the minimum potential energy method to obtain equations of equilibrium by the application of a variational procedure.

Segments near Equators

The strains for thin flat plates with $\bar{u}$ and $\bar{v}$, the displacements in the x and y directions, respectively, and with initial deflection w_0 and additional deflection w are given in rectangular coordinates as follows:

$$\left.\begin{aligned} \epsilon_x &= \bar{u}_{,x} + \tfrac{1}{2}(w_{,x})^2 + w_{,x}w_{0,x} - zw_{,xx} \\ \epsilon_y &= \bar{v}_{,y} + \tfrac{1}{2}(w_{,y})^2 + w_{,y}w_{0,y} - zw_{,yy} \\ \gamma_{xy} &= \bar{u}_{,y} + \bar{v}_{,x} + w_{,x}w_{,y} + w_{,x}w_{0,y} + \\ &\qquad w_{,y}w_{0,x} - 2zw_{,xy} \end{aligned}\right\} \quad (1)$$

For an initial deflection corresponding to segments at the outer equator of a torus of radius r, which has positive Gaussian curvature and which is taken here to have constant curvature $1/a$ in the meridional direction (Fig. 1), w_0 is taken to be

$$w_0 = -(y^2/2r) - (x^2/2a) \quad (2a)$$

The assumption that the initial deflections can be represented in quadratic form is consistent with shallow shell approximations. For segments at the inner equator, the initial deflection has negative Gaussian curvature, and the corresponding equation for w_0 is

$$w_0 = -(y^2/2r) + (x^2/2a) \quad (2b)$$

If Eq. (2a) or Eq. (2b) is substituted in Eqs. (1) and if the following definitions

$$u = \bar{u} \mp w(x/a) \qquad v = \bar{v} - w(y/r) \quad (3)$$

are used, then

$$\left.\begin{aligned} \epsilon_x &= u_{,x} \pm (w/a) + \tfrac{1}{2}(w_{,x})^2 - zw_{,xx} \\ \epsilon_y &= v_{,y} + (w/r) + \tfrac{1}{2}(w_{,y})^2 - zw_{,yy} \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} - 2zw_{,xy} \end{aligned}\right\} \quad (4)$$

where the new u and v can be identified as the tangential displacements of the neutral surface of the shallow shell, and w can now be regarded as the normal displacement. In Eqs. (3) and (4) and in the equations that follow, the convention is used that, when there is a double sign, the upper sign applies to the shell with positive Gaussian curvature and the lower to the shell with negative Gaussian curvature.

The total potential energy of such shells subject to a lateral pressure p is

$$\Pi = \frac{E}{2(1-\mu^2)} \iiint \left(\epsilon_x{}^2 + \epsilon_y{}^2 + 2\mu\epsilon_x\epsilon_y + \frac{1-\mu}{2}\gamma_{xy}{}^2 \right) \times dx\, dy\, dz - p \iint w\, dx\, dy \quad (5)$$

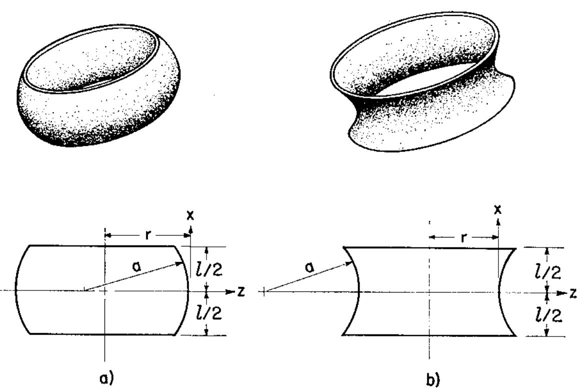

Fig. 1 Configuration for segments near equators.

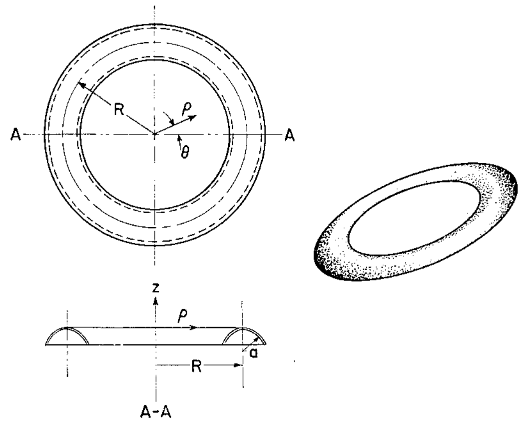

Fig. 2 Configuration for segment near crown.

Integration in the z direction and variation according to the minimum potential energy method leads to the nonlinear differential equations of equilibrium and consistent boundary conditions. The equations of equilibrium are

$$N_{x,x} + N_{xy,y} = 0 \qquad N_{y,y} + N_{xy,x} = 0$$

$$D\nabla^4 w \pm (N_x/a) + (N_y/r) - (N_x w_{,xx} + N_y w_{,yy} + 2N_{xy}w_{,xy}) = p \quad (6)$$

where

$$\left.\begin{aligned} N_x &= (Et/1-\mu^2)(\epsilon_x + \mu\epsilon_y)\big|_{z=0} \\ N_y &= (Et/1-\mu^2)(\epsilon_y + \mu\epsilon_x)\big|_{z=0} \\ N_{xy} &= [Et/2(1+\mu)](\gamma_{xy})\big|_{z=0} \end{aligned}\right\} \quad (7)$$

Note that these equations with $a \to \infty$ are the Donnell large-deflection equations for a cylinder, and if, in addition, $r \to \infty$, then they reduce, of course, to the von Kármán large-deflection equations for a flat plate.

Segments near Crown

The strains for a thin flat plate with an initial deflection w_0 are given in polar coordinates as follows:

$$\left.\begin{aligned} \epsilon_\rho &= u_{,\rho} + \tfrac{1}{2}(w_{,\rho})^2 + w_{,\rho}w_{0,\rho} - zw_{,\rho\rho} \\ \epsilon_\theta &= \frac{1}{\rho}v_{,\theta} + \frac{u}{\rho} + \frac{1}{2\rho^2}(w_{,\theta})^2 + \frac{1}{\rho^2}w_{,\theta}w_{0,\theta} - \\ &\qquad z\left(\frac{1}{\rho^2}w_{,\theta\theta} + \frac{1}{\rho}w_{,\rho}\right) \\ \gamma_{\rho\theta} &= \frac{1}{\rho}u_{,\theta} + v_{,\rho} - \frac{v}{\rho} + \frac{1}{\rho}w_{,\rho}w_{,\theta} + \frac{1}{\rho}w_{,\rho}w_{0,\theta} + \\ &\qquad \frac{1}{\rho}w_{,\theta}w_{0,\rho} - 2z\left(\frac{1}{\rho}w_{,\rho\theta} - \frac{1}{\rho^2}w_{,\theta}\right) \end{aligned}\right\} \quad (8)$$

where w is the additional deflection. For an initial deflection corresponding to segments at the crown of a torus of central radius R where the torus is taken here to have constant meridional curvature $1/a$ (Fig. 2), w_0 is taken to be

$$w_0 = -(\rho - R)^2/2a \quad (9)$$

As before, the assumption that the initial deflections can be represented in quadratic form is consistent with the shallow shell approximation. Thus

$$\left.\begin{aligned} \epsilon_\rho &= u_{,\rho} + \frac{1}{2}(w_{,\rho})^2 - \left(\frac{\rho - R}{a}\right)w_{,\rho} - zw_{,\rho\rho} \\ \epsilon_\theta &= \frac{1}{\rho}v_{,\theta} + \frac{u}{\rho} + \frac{1}{2\rho^2}(w_{,\theta})^2 - z\left(\frac{1}{\rho^2}w_{,\theta\theta} + \frac{1}{\rho}w_{,\rho}\right) \\ \gamma_{\rho\theta} &= \frac{1}{\rho}u_{,\theta} + v_{,\rho} - \frac{v}{\rho} + \frac{1}{\rho}w_{,\rho}w_{,\theta} - \\ &\qquad \left(\frac{\rho - R}{\rho a}\right)w_{,\theta} - 2z\left(\frac{1}{\rho}w_{,\rho\theta} - \frac{1}{\rho^2}w_{,\theta}\right) \end{aligned}\right\} \quad (10)$$

The total potential energy of such a shell subject to a lateral pressure p is

$$\Pi = \frac{E}{2(1-\mu^2)} \iiint \left(\epsilon_\rho^2 + \epsilon_\theta^2 + 2\mu\epsilon_\rho\epsilon_\theta + \frac{1-\mu}{2}\gamma_{\rho\theta}^2\right) \times \rho\, d\rho\, d\theta\, dz - p\iint w\rho\, d\rho\, d\theta \quad (11)$$

Integration in the z direction and variation according to the minimum potential energy method leads to the nonlinear differential equations of equilibrium and consistent boundary conditions. The equations of equilibrium are

$$\left.\begin{aligned} &N_{\rho,\rho} + \frac{1}{\rho}(N_\rho - N_\theta + N_{\rho\theta,\theta}) = 0 \\ &\frac{1}{\rho}(N_{\theta,\theta} + 2N_{\rho\theta}) + N_{\rho\theta,\rho} = 0 \\ &D\nabla^4 w + \frac{N_\rho}{a} + \frac{\rho - R}{a} N_\theta - \left(N_\rho w_{,\rho\rho} + \frac{1}{\rho^2} N_\theta w_{,\theta\theta} + \frac{1}{\rho} N_\theta w_{,\rho} + \frac{2}{\rho} N_{\rho\theta} w_{,\rho\theta} - \frac{2}{\rho^2} N_{\rho\theta} w_{,\theta}\right) = p \end{aligned}\right\} \quad (12)$$

where

$$\left.\begin{aligned} N_\rho &= (Et/1-\mu^2)(\epsilon_\rho + \mu\epsilon_\theta)|_{z=0} \\ N_\theta &= (Et/1-\mu^2)(\epsilon_\theta + \mu\epsilon_\rho)|_{z=0} \\ N_{\rho\theta} &= [Et/2(1+\mu)](\gamma_{\rho\theta})|_{z=0} \end{aligned}\right\} \quad (13)$$

These equations with $R = 0$ reduce to the Marguerre large-deflection shallow spherical cap equations.

Buckling Equations

Buckling equations are derived on the premise of bifurcation behavior with the nonlinear equations just derived used to determine the deformations and stresses prior to buckling and to determine the buckling equations. For the problems considered, the shell, prior to buckling, deforms axisymmetrically. Small changes from this configuration, not necessarily axisymmetric changes, are considered in order to obtain the buckling equations. Buckling equations are also derived, using the assumption that the deflection w is constant prior to buckling.

Segments near Equator

Prior to buckling, for the problems considered, the deformations would be axisymmetric; thus Eqs. (6), with the variables functions of x only, would apply. Thus, for axisymmetric deformations,

$$\begin{gathered} N_{xA,x} = 0 \qquad N_{xyA,x} = 0 \\ Dw_{A,xxxx} \pm (N_{xA}/a) + (N_{yA}/r) - N_{xA}w_{A,xx} = p \end{gathered} \quad (14)$$

where

$$\begin{aligned} N_{xA} &= \frac{Et}{1-\mu^2}\left[u_{A,x} \pm \frac{w_A}{a} + \frac{1}{2}w_{A,x}^2 + \mu\frac{w_A}{r}\right] \\ N_{yA} &= \frac{Et}{1-\mu^2}\left[\frac{w_A}{r} + \mu\left(u_{A,x} \pm \frac{w_A}{a} + \frac{1}{2}w_{A,x}^2\right)\right] \\ N_{xyA} &= [Et/2(1+\mu)]v_{A,x} \end{aligned}$$

To the prebuckling deformations obtainable from these equations and the boundary conditions, small changes that occur during buckling may be added ($u = u_A + u_B$, $v = v_A + v_B$, $w = w_A + w_B$), and the sum should also satisfy Eqs. (6). The preceding relations may be subtracted out after the sums are inserted in Eqs. (6), and terms of higher degree than linear in the buckling displacement may be neglected to give the buckling equations, which follow:

$$\left.\begin{aligned} &N_{xB,x} + N_{xyB,y} = 0 \qquad N_{yB,y} + N_{xyB,x} = 0 \\ &D\nabla^4 w_B \pm \frac{N_{xB}}{a} + \frac{N_{yB}}{r} - (N_{xA}w_{B,xx} + N_{xB}w_{A,xx} + N_{yA}w_{B,yy} + 2N_{xyA}w_{B,xy}) = 0 \end{aligned}\right\} \quad (15)$$

where

$$\begin{aligned} N_{xB} &= \frac{Et}{1-\mu^2}\left[u_{B,x} \pm \frac{w_B}{a} + w_{A,x}w_{B,x} + \mu\left(v_{B,y} + \frac{w_B}{r}\right)\right] \\ N_{yB} &= \frac{Et}{1-\mu^2}\left[v_{B,y} + \frac{w_B}{r} + \mu\left(u_{B,x} \pm \frac{w_B}{a} + w_{A,x}w_{B,x}\right)\right] \\ N_{xyB} &= \frac{Et}{2(1+\mu)}(u_{B,y} + v_{B,x} + w_{A,x}w_{B,y}) \end{aligned}$$

These equations have variable coefficients and would be quite difficult to solve for many cases.

If, instead, the assumption analogous to that of classical cylinder buckling theory is made, that the deflection w is constant prior to buckling (w_A = const), then with the pressure p, a known constant, it may be seen from Eqs. (14) that N_{xA} and N_{yA} must be constant also. Equations from which these constants may be determined for different inplane end conditions are

$$\left.\begin{aligned} N_{xA} &= \frac{Et}{1-\mu^2}\left(u_{A,x} \pm \frac{w_A}{a} + \mu\frac{w_A}{r}\right) \\ N_{yA} &= \frac{Et}{1-\mu^2}\left[\frac{w_A}{r} + \mu\left(u_{A,x} \pm \frac{w_A}{a}\right)\right] \\ &\pm \frac{N_{xA}}{a} + \frac{N_{yA}}{r} = p \end{aligned}\right\} \quad (16)$$

In these expressions, $u_{A,x}$ is also constant, and therefore, for the special case of $u_A = 0$ at both ends, $u_{A,x}$ would be zero. Three different inplane end conditions are considered for the problems solved in this paper and are discussed in a subsequent section.

With w_A a constant, the buckling equations can be written as

$$\left.\begin{aligned} &N_{xB,x} + N_{xyB,y} = 0 \qquad N_{yB,y} + N_{xyB,x} = 0 \\ &D\nabla^4 w_B \pm (N_{xB}/a) + (N_{yB}/r) - (N_{xA}w_{B,xx} + N_{yA}w_{B,yy} + 2N_{xyA}w_{B,xy}) = 0 \end{aligned}\right\} \quad (17)$$

where

$$\begin{aligned} N_{xB} &= \frac{Et}{1-\mu^2}\left[u_{B,x} \pm \frac{w_B}{a} + \mu\left(v_{B,y} + \frac{w_B}{r}\right)\right] \\ N_{yB} &= \frac{Et}{1-\mu^2}\left[v_{B,y} + \frac{w_B}{r} + \mu\left(u_{B,x} \pm \frac{w_B}{a}\right)\right] \\ N_{xyB} &= [Et/2(1+\mu)](u_{B,y} + v_{B,x}) \end{aligned}$$

These buckling equations, obtained through the classical assumption, have constant coefficients and agree with those derived by Becker in Ref. 1; with $a \to \infty$, they agree with the Donnell equations for buckling of a cylinder.

Segments near Crown

Prior to buckling, for the problems considered, the deformations would be axisymmetric; thus Eqs. (12) with the variables functions of ρ, only, would apply:

$$N_{\rho A,\rho} + (1/\rho)(N_{\rho A} - N_{\theta A}) = 0$$

$$(2/\rho)N_{\rho\theta A} + N_{\rho\theta A,\rho} = 0$$

$$\frac{D}{\rho}\left\{\rho\left[\frac{1}{\rho}(\rho w_{A,\rho})_{,\rho}\right]_{,\rho}\right\}_{,\rho} + \frac{N_{\rho A}}{a} + \frac{\rho - R}{\rho a} N_{\theta A} - \left(N_{\rho A}w_{A,\rho\rho} + \frac{1}{\rho}N_{\theta A}w_{A,\rho}\right) = p$$

where

$$N_{\rho A} = \frac{Et}{1-\mu^2}\left[u_{A,\rho} + \frac{1}{2}w_{A,\rho}{}^2 - w_{A,\rho}\left(\frac{\rho - R}{a}\right) + \mu\frac{u_A}{\rho}\right]$$

$$N_{\theta A} = \frac{Et}{1-\mu^2}\left[\frac{u_A}{\rho} + \mu\left(u_{A,\rho} + \frac{1}{2}w_{A,\rho}{}^2 - \frac{\rho - R}{a}w_{A,\rho}\right)\right]$$

$$N_{\rho\theta A} = [Et/2(1+\mu)][v_{A,\rho} - (v_A/\rho)]$$

Proceeding as for segments near the equator, the buckling equations are

$$N_{\rho B,\rho} + (1/\rho)(N_{\rho B} - N_{\theta B} + N_{\rho\theta B,\theta}) = 0$$

$$(1/\rho)(N_{\theta B,\theta} + 2N_{\rho\theta B}) + N_{\rho\theta B,\rho} = 0$$

$$D\nabla^4 w_B + \frac{N_{\rho B}}{a} + \frac{\rho - R}{\rho a}N_{\theta B} - \left(N_{\rho A}w_{B,\rho\rho} + N_{\rho B}w_{A,\rho\rho} + \frac{1}{\rho^2}N_{\theta A}w_{B,\theta\theta} + \frac{1}{\rho}N_{\theta A}w_{B,\rho} + \frac{1}{\rho}N_{\theta B}w_{A,\rho} + \frac{2}{\rho}N_{\rho\theta A}w_{B,\rho\theta} - \frac{2}{\rho^2}N_{\rho\theta A}w_{B,\theta}\right) = 0$$

where

$$N_{\rho B} = \frac{Et}{1-\mu^2}\left[u_{B,\rho} + w_{A,\rho}w_{B,\rho} - \left(\frac{\rho - R}{a}\right)w_{B,\rho} + \mu\left(\frac{1}{\rho}v_{B,\theta} + \frac{u_B}{\rho}\right)\right]$$

$$N_{\theta B} = \frac{Et}{1-\mu^2}\left\{\frac{1}{\rho}v_{B,\theta} + \frac{u_B}{\rho} + \mu\left[u_{B,\rho} + w_{A,\rho}w_{B,\rho} - \left(\frac{\rho - R}{a}\right)w_{B,\rho}\right]\right\}$$

$$N_{\rho\theta B} = \frac{Et}{2(1+\mu)}\left[\frac{1}{\rho}u_{B,\theta} + v_{B,\rho} - \frac{v_B}{\rho} + \frac{1}{\rho}w_{A,\rho}w_{B,\theta} - \left(\frac{\rho - R}{\rho a}\right)w_{B,\theta}\right]$$

For the segment of the toroidal shell near the crown, it is expected that the classical assumption of constant deflection w_A, prior to buckling, will give reasonable stress resultants. With this assumption, the prebuckling stress resultants can be found from

$$N_{\rho A,\rho} + (1/\rho)(N_{\rho A} - N_{\theta A}) = 0 \qquad (2/\rho)N_{\rho\theta A} + N_{\rho\theta A,\rho} = 0$$

$$(N_{\rho A}/a) + [(\rho - R/\rho a)]N_{\theta A} = p$$

The buckling equations then become

$$N_{\rho B,\rho} + (1/\rho)(N_{\rho B} - N_{\theta B} + N_{\rho\theta B,\theta}) = 0$$

$$(1/\rho)(N_{\theta B,\theta} + 2N_{\rho\theta B}) + N_{\rho\theta B,\rho} = 0$$

$$D\nabla^4 w_B + \frac{N_{\rho B}}{a} + \frac{\rho - R}{\rho a}N_{\theta B} - \left(N_{\rho A}w_{B,\rho\rho} + \frac{1}{\rho^2}N_{\theta A}w_{B,\theta\theta} + \frac{1}{\rho}N_{\theta A}w_{B,\rho} + \frac{2}{\rho}N_{\rho\theta A}w_{B,\rho\theta} - \frac{2}{\rho^2}N_{\rho\theta A}w_{B,\theta}\right) = 0$$

where

$$N_{\rho B} = \frac{Et}{1-\mu^2}\left[u_{B,\rho} - \left(\frac{\rho - R}{a}\right)w_{B,\rho} + \mu\left(\frac{1}{\rho}v_{B,\theta} + \frac{u_B}{\rho}\right)\right]$$

$$N_{\theta B} = \frac{Et}{1-\mu^2}\left\{\frac{1}{\rho}v_{B,\theta} + \frac{u_B}{\rho} + \mu\left[u_{B,\rho} - \left(\frac{\rho - R}{a}\right)w_{B,\rho}\right]\right\}$$

$$N_{\rho\theta B} = \frac{Et}{2(1+\mu)}\left[\frac{1}{\rho}u_{B,\theta} + v_{B,\rho} - \frac{v_B}{\rho} - \left(\frac{\rho - R}{\rho a}\right)w_{B,\theta}\right]$$

Solutions for Segments near Equators

Closed-form solutions to the buckling equations just derived are now presented for simply supported segments of toroidal shells near the equators under pressure loading with various inplane support conditions. The shells considered extend completely around the equator, and the equator lies at their midlength (Fig. 1). Shells of both positive and negative Gaussian curvature are considered. The inplane support conditions considered are listed below together with the N_{xA} and N_{yA} determined (on the basis of the classical assumption that w_a is a constant) from Eqs. (16):

$$\left.\begin{aligned}
&N_{xA} = 0 \qquad N_{yA} = pr \qquad \text{zero edge load}\\
&N_{xA} = (pr/2) \qquad N_{yA} = pr[1 \mp (r/2a)]\\
&\qquad\qquad \text{hydrostatic pressure loaded}\\
&N_{xA} = \frac{p[(\mu/r) \pm (1/a)]}{(1/r^2) \pm (2\mu/ra) + (1/a^2)}\\
&N_{yA} = \frac{p[(1/r) \pm (\mu/a)]}{(1/r^2) \pm (2\mu/ra) + (1/a^2)}\\
&\qquad\qquad \text{zero edge displacement}
\end{aligned}\right\} \tag{18}$$

With no applied shear stress, $N_{xyA} = 0$.

The buckling equations [Eqs. (17)] obtained through the classical assumption may be written in terms of u_B, v_B, w_B as (dropping the subscript B)

$$\left.\begin{aligned}
&u_{,xx} + \frac{1-\mu}{2}u_{,yy} + \frac{1+\mu}{2}v_{,xy} + \left(\frac{\mu}{r} \pm \frac{1}{a}\right)w_{,x} = 0\\
&\frac{1+\mu}{2}u_{,xy} + v_{,yy} + \frac{1-\mu}{2}v_{,xx} + \left(\frac{1}{r} \pm \frac{\mu}{a}\right)w_{,y} = 0\\
&D\nabla^4 w + \frac{Et}{1-\mu^2}\left[\left(\frac{\mu}{r} \pm \frac{1}{a}\right)u_{,x} + \left(\frac{1}{r} \pm \frac{\mu}{a}\right)v_{,y} + \left(\frac{1}{a^2} \pm \frac{2\mu}{ar} + \frac{1}{r^2}\right)w\right] -\\
&\qquad (N_{xA}w_{,xx} + N_{yA}w_{,yy} + 2N_{xyA}w_{,xy}) = 0
\end{aligned}\right\} \tag{19}$$

With the origin now taken along the lower edge, the simple support boundary conditions on the buckling displacements at $x = 0, l$ are

$$w = w_{,xx} = v = N_x = 0 \tag{20}$$

Solutions that satisfy the boundary conditions, Eq. (20), and the differential equations [Eqs. (19)] for any one of the three inplane edge conditions are

$$\left.\begin{aligned}
u &= A\cos(m\pi x/l)\sin(ny/r)\\
v &= B\sin(m\pi x/l)\cos(ny/r)\\
w &= C\sin(m\pi x/l)\sin(ny/r)
\end{aligned}\right\} \tag{21}$$

For the problems considered here, $m = 1$ applies, since it gives the lowest buckling load. The buckling loads found from this solution as a function of the number of circumferential waves n are given by the following relations:

$$\left.\begin{aligned}
&k = \frac{(1+\beta^2)^4 + (12Z^2/\pi^4)(1 \pm \alpha\beta^2)^2}{\beta^2(1+\beta^2)^2}\\
&\qquad\qquad \text{for zero edge loading}\\
&k = \frac{(1+\beta^2)^4 + (12Z^2/\pi^4)(1 \pm \alpha\beta^2)^2}{\frac{1}{2}(1 + 2\beta^2 \mp \alpha\beta^2)(1+\beta^2)^2}\\
&\qquad\qquad \text{for hydrostatic pressure loading}\\
&k = \frac{(1+\beta^2)^4 + (12Z^2/\pi^4)(1 \pm \alpha\beta^2)^2}{[\mu \pm \alpha + (1 \pm \mu\alpha)\beta^2/\alpha^2 \pm 2\mu\alpha + 1/(1+\beta^2)^2}\\
&\qquad\qquad \text{for zero edge displacement}
\end{aligned}\right\} \tag{22}$$

where $k = -(prl^2/D\pi^2)$, $\beta = (nl/\pi r)$, $\alpha = r/a$, and $Z = (l^2/rt)(1 - \mu^2)^{1/2}$. The buckling pressures are obtained from Eqs. (22) when k is minimized with respect to allowable changes in β for given Z and α.

The buckling pressure coefficients have been calculated for Poisson's ratio $\mu = \frac{1}{3}$, and the results of the calculations are plotted in Figs. 3–6. For the shell of positive Gaussian curvature under external pressure, the results for higher values of Z lie along a straight line as shown on the logarithmic plot. Simple results for the critical pressure obtained analytically (assuming β large) for any value of Poisson's ratio are presented below for these straight line regions:

$$-p = \frac{1}{[3(1-\mu^2)]^{1/2}}\frac{Et^2}{ar} \quad \text{for zero edge loading}$$

$$-p = \frac{2}{(2-\alpha)[3(1-\mu^2)]^{1/2}}\frac{Et^2}{ar} \quad \text{for hydrostatic pressure loading}$$

$$-p = \frac{1+2\mu\alpha+\alpha^2}{(1+\alpha)[3(1-\mu^2)]^{1/2}}\frac{Et^2}{ar} \quad \text{for zero edge displacement}$$

Estimate of External Buckling Pressure for Complete Torus

Prebuckling values for the stress resultants for the pressure loading of a complete circular torus analogous to those obtained for the shallow shell [see Eqs. (18)] are

$$N_{xA} = pa[1 - (a/2r)] \qquad N_{yA} = pa/2 \tag{23}$$

and proceeding with these values as for the shallow shell leads to

$$k = \frac{(1+\beta^2)^4 + (12Z^2/\pi^4)(1+\alpha\beta^2)^2}{(1+\beta^2)^2\left(\frac{1}{\alpha} - \frac{1}{2\alpha^2} + \frac{1}{2\alpha}\beta^2\right)} \tag{24}$$

where l in β, k, and Z must be interpreted as the buckle axial length. By taking β large, the buckling pressure can be estimated to be

$$-p = \{2/[3(1-\mu^2)]^{1/2}\}(Et^2/a^2) \tag{25}$$

This expression is probably valid only when r/a is not much greater than 2. This result indicates that the torus buckles when the circumferential stress reaches $0.6(Et/a)$. The classical results show that a cylinder of radius a in axial compression buckles when the axial stress reaches $0.6(Et/a)$, and the circumferential stress is zero. The sphere of radius a under hydrostatic pressure buckles when the stress (in any direction) reaches $0.6(Et/a)$. However, the torus under hydrostatic pressure buckles when the circumferential stress reaches $0.6(Et/a)$, even though the meridional stress exceeds this value.

Discussion of Results

Nonlinear equations for segments of toroidal shells near the equator and near the crown have been derived from flat-plate equations by including appropriate prescribed initial deflections. Buckling equations have been derived for both kinds of segments. Utilizing the classical assumption of constant deflections prior to buckling, buckling equations with constant coefficients have been obtained for segments near the equator. These equations have been solved in closed form for pressure loading of simply supported segments with both positive and negative Gaussian curvature, having three different inplane edge conditions: 1) zero edge load, 2) hydrostatic pressure loaded, and 3) zero edge displacement.

Results obtained for the case of external pressure buckling with zero edge load (lateral pressure) are presented in Fig. 3. For a given value of the curvature parameter Z associated with the girth of the shell, the external pressure required for buckling increases significantly over the buckling pressure for the cylinder ($r/a = 0$) as the curvature in the meridional direction is increased to form a shell of positive Gaussian curvature. The corresponding external pressure decreases significantly as the curvature in the meridional direction is increased to form a shell of negative Gaussian curvature. For higher values of Z, the curve for the cylinder ($r/a = 0$) has a slope of $\frac{1}{2}$. For the spherical segment ($r/a = 1$), the curve has a slope of unity. And for the corresponding shell ($r/a = 1$) of negative Gaussian curvature, the curve has a slope of zero; thus the curvature contributes little to the strength. Both the shell of positive and the shell of negative curvature do not buckle under internal pressure for this inplane edge condition.

Results obtained for the case of external hydrostatic pressure buckling are presented in Fig. 4. These results follow trends similar to the lateral pressure results, and again both the shell of positive and negative Gaussian curvature do not buckle under internal pressure. Again significant increases in external pressure required for buckling are available for shells of positive Gaussian curvature over cylinders of the same curvature parameter Z, and significant decreases in buckling pressure occur for shells of negative Gaussian curvature over cylinders of the same Z.

Results obtained for the cases of external and internal pressure buckling, respectively, of shells with zero edge displacement are presented in Figs. 5 and 6. In the case of external pressure, as shown in Fig. 5, significant increases in the pressure required for buckling are available for shells of both

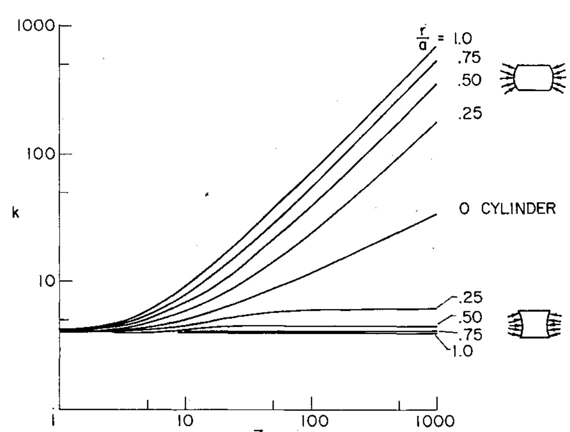

Fig. 3 Buckling of toroidal segments under external lateral pressure.

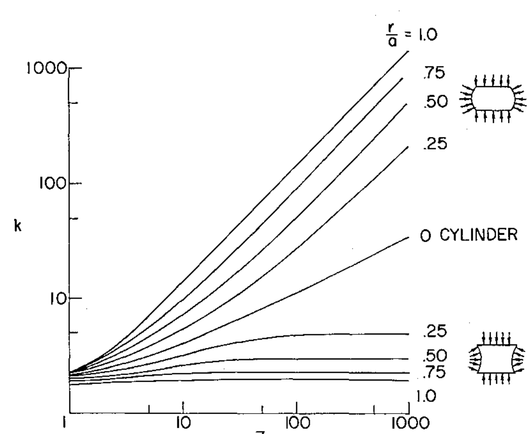

Fig. 4 Buckling of toroidal segments under external hydrostatic pressure.

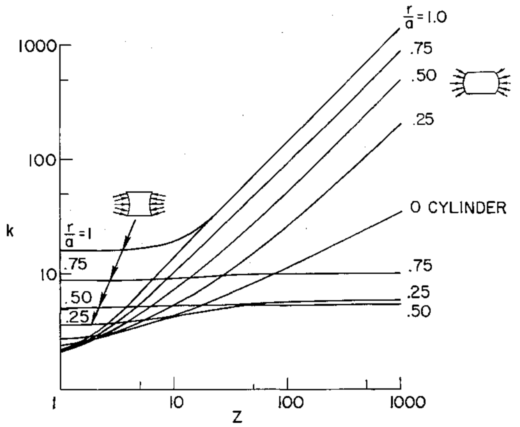

Fig. 5 Buckling of toroidal segments with zero edge displacement under external lateral pressure.

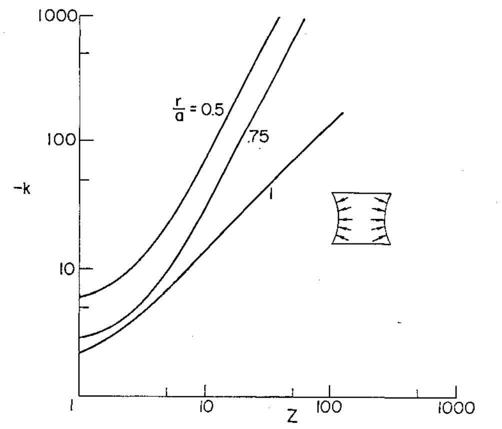

Fig. 6 Buckling of toroidal segments with zero edge displacement under internal lateral pressure.

positive and negative Gaussian curvature over that for a cylinder at the same value of the curvature parameter Z. Because of the $u = 0$ condition, tensile stresses develop at the edges that tend to stabilize the shell of negative curvature as the ratio of radii r/a increases. The shell with negative Gaussian curvature also buckles under internal pressure with this zero edge displacement condition (Fig. 6) provided that the ratio r/a is greater than Poisson's ratio μ. As r/a increases for given Z, compressive stresses at the edges increase and cause a decrease in the buckling pressure.

Concluding Remarks

The present analysis starts with accepted (von Kármán) nonlinear flat-plate strains, including initial deflections, and in a consistent and straightforward manner derives nonlinear shallow shell equations from which buckling equations are determined. Nonlinear equations and buckling equations for other shallow shells such as conical frustums away from the apex and segments of toroidal shells away from the equator or crown may be derived in a similar manner.

Buckling pressures have been obtained in chart form for simply supported toroidal segments near the equators having both positive and negative Gaussian curvature. Three inplane edge conditions are considered. Limiting values of the buckling pressure for shells of large Z having positive Gaussian curvature are given in equation form. An estimate based on this shallow shell theory of the buckling pressure of a complete torus is also given in equation form.

The present theory is limited by the shallow shell approximation that, however, may not be too serious for deeper shell buckling analysis. This conjecture that shallow shell theory may be used to analyze some deep shells is based on the consideration that shells buckle first where the curvature is most shallow. This consideration was used in estimating the buckling pressure for the complete torus.

Another limitation of the present results exists because of the disagreement between some shell buckling solutions for perfect shells and experiment. For the cylinder under external pressure, there is good agreement between theory and experiment. However, for the spherical segment it is expected that agreement between theory and experiment will not be nearly as good as for the cylinder, judging from buckling results for the spherical cap. Similar limitations probably arise for the buckling of other toroidal segments and for the complete torus. Thus, for design purposes, the present results may only serve as a guide by specifying the buckling pressure for a perfect shell.

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